Topic - Problem based on the Determination of the nth Derivative with the help of Leibnitz's Theorem

Class - B.Sc.(Hons)

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1. If. $y = x^2 e^{ax}$

Solution :→ We have

$$y = x^2 e^{ax}$$
$$= (e^{ax})\, x^2$$

Say $u = e^{ax}$ $\quad v = x^2$

$\Rightarrow u_n = a^n e^{ax}$ $\quad v_1 = 2x$

$u_{n-1} = a^{n-1} e^{ax}$ $\quad v_2 = 2$

$u_{n-2} = a^{n-2} e^{ax}$ $\quad v_3 = 0 = \cdots = v_n$

Now Differentiating n times with respect to x by Leibnitz's Theorem

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \cdots + {}^nC_n u v_n.$$

$$ {}^nC_0 [a^n e^{ax} x^2] + {}^nC_1 (a^{n-1} e^{ax})\, 2x + {}^nC_2\, a^{n-2} e^{ax}\, 2 + 0 + \cdots + 0$$

$$= a^n e^{ax} \cdot x^2 + 2n\, a^{n-1} e^{ax} \cdot x + \frac{n(n-1)}{1\cdot 2} \cdot 2 e^{ax} \cdot a^{n-2} + 0 + 0 \cdots$$

$$\therefore y_n = a^n e^{ax} x^2 + 2nx\, e^{ax} a^{n-1} + n(n-1)\, e^{ax} \cdot a^{n-2}$$
$$= a^{n-2} e^{ax} [a^2 x^2 + 2nx a + n(n-1)]. \text{ Proved}$$

2. Find y_n when $y = x^2 \sin x$

By the question

$$y = x^2 \sin x$$
$$= \sin x \cdot x^2$$

Say $u = \sin x$ $\qquad v = x^2$

$$v_1 = 2x$$
$$u_n = \sin\left(\frac{n\pi}{2} + x\right) \qquad v_2 = 2$$
$$v_3 = v_4 \cdots = 0$$
$$u_{n-1} = \sin\left[\frac{(n-1)}{2}\pi + x\right]$$
$$u_{n-2} = \sin\left[\frac{n-2}{2}\pi + x\right]$$

Differentiating both sides n times with respect to x by Leibnitz's theorem

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots$$
$$\cdots + {}^nC_n u v_n$$
$$= \sin\left(\frac{n\pi}{2} + x\right) \cdot x^2 + n \cdot \sin\left(\frac{n-1}{2}\pi + x\right) \cdot 2x$$
$$+ \frac{n(n-1)}{1 \cdot 2} \sin\left(\frac{n-2}{2}\pi + x\right) \cdot 2 + 0 + 0 \cdots$$
$$= x^2 \sin\left(\frac{n\pi}{2} + x\right) + 2nx \sin\left(\frac{n-1}{2}\pi + x\right)$$
$$+ n(n-1) \sin\left(\frac{n-2}{2}\pi + x\right)$$

3. If $y = x^{n-1} \log x$

prove that

$$y_n = \frac{\lfloor n-1}{x}$$

Solution

We have

$$y = x^{n-1} \log x$$

Diff. with respect to x

$$y_1 = (n-1)x^{n-2}\log x + x^{n-1}\cdot\frac{1}{x}$$

$$= (n-1)x^{n-2}\log x + x^{n-1}\cdot\frac{1}{x}$$

$$\Rightarrow xy = (n-1)x^{n-1}\log x + x^{n-1}$$

$$= (n-1)y + x^{n-1}$$

Differentiating (n-1) times with respect to x by using Leibnitz

$$xy_n + {}^{n-1}C_1\, y_{n-1}\cdot 1 = (n-1)y_{n-1} + \lfloor n-1$$

$$\Rightarrow xy_n + \cancel{(n-1)y_{n-1}} = \cancel{(n-1)y_{n-1}} + \lfloor n-1$$

$$\Rightarrow y_n = \frac{\lfloor n-1}{x}$$

Hence the required result.